5.3 Diagonalization (*)

Consider

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Compute the product: $P^{-1}AP$.

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D \text{ (diagonal matrix)}$$

It can be shown (DO It!) that

$\lambda_1 = 0$ is an eigenvalue with corresp. eigenvector $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ " " " " " eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(*) Remark: In this section, we will consider matrices with real eigenvalues only. However, they may be repeated.

This implies

$$P = [\vec{u}\ \vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$D = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def.

A matrix $A_{n\times n}$ such that there exists P invertible matrix satisfying

$$P^{-1}AP = D \text{ (diagonal matrix)} \qquad (2.1)$$

is said to be diagonalizable.

From the previous example we arrive to two important conclusion

1) A good candidate for an invertible matrix P satisfying (2.1) is a matrix whose columns are n linearly independent eigenvectors.

2) The diagonal matrix D has the eigenvalues along the diagonal.

Thm 5 Diagonalization Thm.

A matrix $A_{n\times n}$ is diagonalizable if and only if A has n linearly independent eigenvectors.

$$P^{-1}AP = D$$

where Columns of P are linearly independent eigenvectors and D is a diagonal matrix with the Corresponding eigenvalues along the diagonal.

Before proving this theorem observe the following:

If $P = [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]$, $A_{n\times n}$, and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & 0 \\ & 0 & & \lambda_n \end{bmatrix}$$
(diagonal Matrix)

Then,

$$AP = A[\vec{v}_1 \dots \vec{v}_n] = [A\vec{v}_1\ A\vec{v}_2 \dots A\vec{v}_n] \qquad (3.1)$$

and

$$PD = [\vec{v}_1 \cdots \vec{v}_n]D = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2 \dots \lambda_n\vec{v}_n] \qquad (3.2)$$

Proof thm 5.-

$A_{n\times n}$ diagonalizable $\Longleftrightarrow$ There exists $P=[\vec{v}_1\, \vec{v}_2 \dots \vec{v}_n]$ invertible such that $P^{-1}AP=D=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} \Longleftrightarrow$ There exists P invertible such that $AP=PD$

$\overset{(3.1),(3.2)}{\Longleftrightarrow}$ $[A\vec{v}_1\; A\vec{v}_2 \dots A\vec{v}_n]=[\lambda_1\vec{v}_1\; \lambda_2\vec{v}_2 \dots \lambda_n\vec{v}_n]$ and $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ is linearly independent.

$\Longleftrightarrow$ $A\vec{v}_1=\lambda_1\vec{v}_1,\ A\vec{v}_2=\lambda_2\vec{v}_2,\dots,\ A\vec{v}_n=\lambda_n\vec{v}_n$ and $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ is lin. independent

$\Longleftrightarrow$ A has n linearly independent eigenvectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$, with corresponding eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$ (possibly repeated).

Thm 5 suggests a <u>method to diagonalize a matrix A</u> if possible. (Review previous example).

1) Find eigenvalues of A: $\lambda_1, \lambda_2, .., \lambda_n$ (possibly repeated)

2) Find n linearly independent eigenvectors of A, $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_n\}$
(if possible). Otherwise A is not diagonalizable according to thm 5.

3) Construct P from eigenvectors obtained in step 2.
More precisely, define

$$P = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$$

Clearly, P is invertible because the set of eigenvectors $S = \{\vec{v}_1, .., \vec{v}_n\}$ is lin. indep.

and

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where $\lambda_1, .., \lambda_n$ are the eigenvalues of A corresponding to $\vec{v}_1, .., \vec{v}_n$ (possibly repeated).

<u>Thm 2</u> (Section 5.1)

If $A_{n\times n}$ matrix is such that

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, \quad A\vec{v}_r = \lambda_r \vec{v}_r, \quad \underbrace{\vec{v}_i \neq \vec{0}}_{\text{eigenvectors}}, \; i=1,2,..$$

and $\lambda_1, .., \lambda_r$ are distinct eigenvalues,

then the set $\{\vec{v}_1, .., \vec{v}_r\}$ is linearly independent.

<u>Proof.-</u>

Want to prove

If $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdot\cdot + c_r\vec{v}_r = \vec{0} \Rightarrow c_1 = 0, .., c_r = 0.$

So, start assuming

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0} \qquad (6.1)$$

And multiply by A both sides

$$A(c_1\vec{v}_1 + \cdot\cdot + c_r\vec{v}_r) = A\vec{0} = \vec{0}$$

Using matrix operations and the hypothesis $\underset{\text{eigenval.}}{\lambda\text{'s}}$ and $\underset{\text{eigenvect.}}{\vec{v}\text{'s}}$

$$c_1 A\vec{v}_1 + \cdot + c_r A\vec{v}_r = \vec{0} \Leftrightarrow c_1\lambda_1\vec{v}_1 + \cdot\cdot + c_r\lambda_r\vec{v}_r = \vec{0}$$

multiplying $\boxed{\text{Equ. (6.1)} \times \lambda_1}$ and subtracting $\quad - c_1\lambda_1\vec{v}_1 - \cdots - c_r\lambda_1\vec{v}_r = 0$

$$c_2(\lambda_2 - \lambda_1)\vec{v}_1 + \cdot\cdot + c_r(\lambda_r - \lambda_1)\vec{v}_r = \vec{0}$$

$\Rightarrow \boxed{c_2 = 0, .., c_r = 0}$, Since $\lambda_2 \neq \lambda_1, .., \lambda_r \neq \lambda_1$ and $\vec{v}_2, .., \vec{v}_r \neq \vec{0}$.

Substitution into (6.1) $\Rightarrow \boxed{c_1 = 0.}$ ✓

Theorem 6

Any $A_{n\times n}$ matrix with n distinct eigenvalues: $\lambda_1, \ldots, \lambda_n$ is diagonalizable.

Proof.- Trivial consequence of thm 2 and thm 5.

If A has n distinct eigenvalues $\overset{\text{Thm 2}}{\Longrightarrow}$ The set $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of the eigenvectors corresponding to the n distinct eigenvalues is linearly independent.

$\overset{\text{Thm 5}}{\Longrightarrow}$ A is diagonalizable.

Why the reciprocal statement:

A diagonalizable $\Longrightarrow$ A has n distinct eigenvalues

is not TRUE? See our previous example!

Consider

Ex #13 (5.1) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Determine if A is diagonalizable or not.

1) First compute eigenvalues of A

$$|A - \lambda I| = 0 \iff \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\iff (4-\lambda)(1-\lambda)(1-\lambda) - [-2(1-\lambda)] = 0$$

$$\iff (1-\lambda)[(4-\lambda)(1-\lambda)+2] = (1-\lambda)[\lambda^2 - 5\lambda + \underbrace{4+2}_{6}]$$

$$= (1-\lambda)[(\lambda-3)(\lambda-2)] = 0$$

$$\iff \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$$

Since $A_{3\times3}$ has 3 distinct eigenvalues by thm 6

<u>A is diagonalizable.</u> ✓

If we also would like to find a matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We need to find the eigenvectors associated to each eigenvalue

For $\lambda_1 = 1$ $\quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

For $\lambda_2 = 2$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda_3 = 3$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Then

$$P = \underbrace{\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{order of the columns is important}}.$$

Why we don't need to verify that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent?

5.2 Finding Eigenvalues and Corresponding Eigenvectors

Consider

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

The eigenvalues of A are the roots of the Characteristic Polynomial:

$$P(\lambda) = \det(A - \lambda I)$$

In our case,

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} =$$

$$\begin{bmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{matrix} -\lambda & -1 \\ 1 & 2-\lambda \\ -1 & -1 \end{matrix} = (-\lambda)(2-\lambda)(-\lambda) + (-1)(1)(-1)$$

$$+ (-1)(1)(-1) - \left[(-1)(2-\lambda)(-1) + (-1)(1)(-\lambda) + (-\lambda)(1)(-1) \right]$$

$$= \lambda^2(2-\lambda) + 1 + 1 - \left((2-\lambda) + 2\lambda \right) = \lambda^2(2-\lambda) + \cancel{2} - \cancel{2} + \lambda - 2\lambda$$

$$= \lambda^2(2-\lambda) - \lambda = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1)$$

$$= -\lambda(\lambda - 1)^2$$

$\lambda_1 = 0$, $\lambda_2 = 1$ (repeated twice) or $\lambda_2 = 1$ has multiplicity 2

Eigenvectors:

$\lambda_1 = 0$ Solve the equation

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$x_1 = x_3$
$x_2 = -x_3$
x_3 free

$$\boxed{\vec{x} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

eigenvector corresponding $\lambda_1 = 0$.

$$A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \vec{0}.$$

$\lambda_2 = 1$

$$(A - 1.I)\,\vec{x} = 0$$

$$[A - 1.I \quad \begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 = -x_2 - x_3$ or $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

x_2 free
x_3 free

Then for

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Eeigenvalues and Eigenvectors are

1) $\lambda_1 = 0 \quad u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and any multiple $\vec{w} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \alpha \vec{u}$

2) $\lambda_2 = 1 \quad \vec{V}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and

Any linear combination

$$\vec{W} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace corresponding to $\lambda_1 = 0$ is $B_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

A basis for the eigenspace corresp. to $\lambda_2 = 1$ is $B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Consider now the matrix P whose column vectors are the vectors in B_1 and B_2.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let's compute its inverse

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right], \quad \text{So } P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

This computation also implies that the three eigenvectors of A:

$u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.